DEPARTMENT OF MATHEMATICS

MAT391 ENGINEERING MATHEMATICS III

WEEK 2: LECTURE NOTE

INVERSE LAPLACE TRANSFORMS

The symbol $L^{-1}[F(s)]$ denotes a causal function $f(t)$ whose Laplace transform is $F(s)$; that is,

$$\text{if} \quad L[f(t)] = F(s) \quad \text{then} \quad f(t) = L^{-1}[F(s)].$$

This correspondence between the functions $F(s)$ and $f(t)$ is called **the inverse Laplace transformation,** $f(t)$ being the inverse transform of $F(s)$, and L^{-1} being referred to as **the inverse Laplace transform operator**.

The Laplace transform $F(s)$ only determines the behaviour of $f(t)$ for $t \geq 0$. Thus $L^{-1}[F(s)] = f(t)$ only for $t \geq 0$. When writing $L^{-1}[F(s)] = f(t)$, it is assumed that $t \geq 0$.

Example 1. *Since* $L[e^{at}] = \dfrac{1}{s-a}$ *it follows that*

$$L^{-1}[\frac{1}{s-a}] = e^{at}.$$

We frequently need to determine the inverse transform of a rational function of the form $\dfrac{p(s)}{q(s)}$, where $p(s)$ and $q(s)$ are polynomials in s. In such cases the procedure is first to resolve the function into **partial fractions** and then to use the table of transforms.

Rules of Partial Fractions

1. The numerator must of lower degree than the denominator.
2. Factorize the denominator into its prime factors. These determine the shapes of the partial fractions.
3. A linear factor $(s+a)$ gives a partial fraction $\dfrac{A}{s+a}$ where A is a constant to be determined.
4. A repeated factor $(s+a)^2$ gives $\dfrac{A}{s+a} + \dfrac{B}{(s+a)^2}$.
5. $(s+a)^3$ gives $\dfrac{A}{s+a} + \dfrac{B}{(s+a)^2} + \dfrac{C}{(s+a)^3}$.
6. An irreducible quadratic factor s^2+as+b gives $\dfrac{As+B}{s^2+as+b}$.
7. A repeated quadratic factor $(s^2+as+b)^2$ gives $\dfrac{As+B}{s^2+as+b} + \dfrac{Cs+D}{(s^2+as+b)^2}$.

Example 2. *Determine inverse Laplace transform of* $F(s) = \dfrac{3s+1}{s^2-s-6}$.

Solution. Note that

$$\frac{3s+1}{s^2-s-6} = \frac{3s+1}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = \frac{(A+B)s+2A-3B}{s^2-s-6}$$

so that we have a simultaneous equation $\begin{cases} A+B=3 \\ 2A-3B=1. \end{cases}$ Solving for A and B we get $A=2$ and $B=1$. Therefore,

$$\frac{3s+1}{s^2-s-6}=\frac{2}{s-3}+\frac{1}{s+2}=L[2e^{3t}]+L[e^{-2t}]=L[2e^{3t}+e^{-2t}],$$

so that

$$L^{-1}[F(s)]=L^{-1}[\frac{3s+1}{s^2-s-6}]=2e^{3t}+e^{-2t}=f(t).$$

Example 3. *Determine inverse Laplace transform of* $F(s)=\dfrac{3s^2-4s+11}{(s+3)(s-2)^2}$.

Solution. Note that

$$\begin{aligned}\frac{3s^2-4s+11}{(s+3)(s-2)^2}&=\frac{A}{s+3}+\frac{B}{s-2}+\frac{C}{(s-2)^2}\\&=\frac{A(s-2)^2+B(s+3)(s-2)+C(s+3)}{(s+3)(s-2)^2}\\&=\frac{(A+B)s^2+(-4A+B+c)s+(4A-6B+3C)}{(s+3)(s-2)^2}\end{aligned}$$

so that we have a system of equations $\begin{cases} A+B=3 \\ -4A+B+C=-4 \\ 4A-6B+3C=11 \end{cases}$ written in matrix form as

$$\begin{pmatrix} 1 & 1 & 0 \\ -4 & 1 & 1 \\ 4 & -6 & 3 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}=\begin{pmatrix} 3 \\ -4 \\ 11 \end{pmatrix}$$

Solving for A, B and C we get

$$A=2, \quad B=1, \quad \text{and} \quad C=3.$$

Therefore,

$$\frac{3s^2-4s+11}{(s+3)(s-2)^2}=\frac{2}{s+3}+\frac{1}{s-2}+\frac{3}{(s-2)^2},$$

so that

$$\begin{aligned}L^{-1}[F(s)]&=L^{-1}[\frac{2}{s+3}]+L^{-1}[\frac{1}{s-2}]+L^{-1}[\frac{3}{(s-2)^2}]\\&=2e^{-3t}+e^{2t}+3te^{2t}+3te^{2t}=f(t).\end{aligned}$$

Example 4. *Determine inverse Laplace transform of* $F(s)=\dfrac{4s^2-5s+6}{(s+1)(s^2+4)}$.

Solution. Note that

$$\begin{aligned}\frac{4s^2-5s+6}{(s+1)(s^2+4)}&=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}=\frac{A(s^2+4)+(s+1)(Bs+C)}{(s+1)(s^2+4)}\\&=\frac{(A+B)s^2+(B+c)s+(4A+C)}{(s+1)(s^2+4)}\end{aligned}$$

so that we have a system of equations

$$\begin{cases} A+B=4 \\ B+C=-5 \\ 4A+C=6 \end{cases}$$

which is written in matrix form as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$$

Solving for A, B and C we get

$$A = 3, \quad B = 1, \quad \text{and} \quad C = -6.$$

Therefore,

$$\frac{4s^2 - 5s + 6}{(s+1)(s^2+4)} = \frac{3}{s+1} + \frac{s-6}{s^2+4} = \frac{3}{s+1} + \frac{s}{s^2+4} - \frac{6}{s^2+4},$$

so that

$$\begin{aligned} f(t) = L^{-1}[F(s)] &= L^{-1}[\frac{3}{s+1}] + L^{-1}[\frac{s}{s^2+4}] - 3L^{-1}[\frac{2}{s^2+4}] \\ &= 3e^{-t} + \cos 2t - 3\sin 2t. \end{aligned}$$

Example 5. *Determine inverse Laplace transform of* $F(s) = \dfrac{s+17}{(s-1)(s+2)(s-3)}$.

Solution. Note that

$$L^{-1}[\frac{s+17}{(s-1)(s+2)(s-3)}] = L^{-1}[\frac{A}{s-1}] + L^{-1}[\frac{B}{s+2}] + L^{-1}[\frac{C}{s-3}]$$

By the Cover Up Rule:

$$\begin{aligned} A &= \frac{1+17}{(1+2)(1-3)} = -3 \qquad (s=1) \\ B &= \frac{-2+17}{(-2-1)(-2-3)} = 1 \qquad (s=-2) \\ C &= \frac{3+17}{(3-1)(3+2)} = 2 \qquad (s=3) \end{aligned}$$

Therefore,

$$\begin{aligned} f(t) &= L^{-1}[F(s)] \\ &= L^{-1}[\frac{s+17}{(s-1)(s+2)(s-3)}] \\ &= L^{-1}[\frac{-3}{s-1}] + L^{-1}[\frac{1}{s+2}] + L^{-1}[\frac{2}{s-3}] \\ &= -3e^{t} + e^{-2t} + 2e^{3t}. \end{aligned}$$

Example 6. *Determine inverse Laplace transform of*

$$F(s) = \frac{s+5}{s^2+2s+5}.$$

Solution. *By completing the Squares we get*

$$\begin{aligned} \frac{s+5}{s^2+2s+5} &= \frac{s+5}{s^2+2s+1-1+5} \\ &= \frac{s+5}{(s+1)^2+4} \\ &= \frac{s+1}{(s+1)^2+4} + 2[\frac{2}{(s+1)^2+4}] \end{aligned}$$

Therefore,

$$\begin{aligned} f(t) &= L^{-1}[F(s)] \\ &= L^{-1}[\frac{s+5}{s^2+2s+5}] \\ &= L^{-1}[\frac{s+1}{(s+1)^2+4}] + 2L^{-1}[\frac{2}{(s+1)^2+4}] \\ &= e^{-t}\cos 2t + 2e^{-t}\sin 2t \\ &= e^{-t}(\cos 2t + 2\sin 2t) \end{aligned}$$

Table of Inverse Laplace transforms

$F(s)$	$f(t) = L^{-1}[F(s)]$
$\frac{a}{s}$	a
$\frac{1}{s^n} \quad n = 1, 2, 3, ...$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{n!}{s^{n+1}} \quad n = 1, 2, 3, ...$	t^n
$\frac{1}{s-a}$	e^{at}
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{a}{s^2-a^2}$	$\sinh at$

Solution of differential equations by Laplace transforms

Definition 7 (**Growth Restriction/Growth Condition**). *We say that a function $f(t)$ defined for $t \geq 0$ satisfy the* ***growth restriction (growth condition)*** *if there exist constants M and k such that*

$$|f(t)| \leq Me^{kt} \quad \text{for all} \quad t \geq 0.$$

Theorem 8 (**Transforms of derivatives**). *Let f, f', f'', ..., f^{n-1} be continuous for all $t \geq 0$ and satisfy the growth condition. Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Then the Laplace transform of f^n satisfies*

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - ... - sf^{(n-2)}(0) - f^{n-1}(0).$$

If $L[f(t)] = F(s)$, then it follows from the definition that

1. $L[f'(t)] = sF(s) - f(0)$.
2. $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$.
3. $L[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$.

Theorem 9 (**Laplace Transforms of integrals**). *Let $F(s)$ denote the Laplace transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$ and satisfies a growth restriction $|f(t)| \leq Me^{kt}$ for all $t \geq 0$. Then, for $s > 0$, $s > k$, and $t > 0$,*

$$L\left[\int_0^t f(\eta)d\eta\right] = \frac{1}{s}F(s).$$

It follows from Theorem 3 that

$$\int_0^t f(\eta)d\eta = L^{-1}\left[\frac{1}{s}F(s)\right]$$

Example 10 (Laplace Transforms of integrals).

$$L\left[\int_0^t (\eta^3 + \sin 2\eta)d\eta\right] = \frac{1}{s}L\left[t^3 + \sin 2t\right] = \frac{1}{s}\left[\frac{3!}{s^4} + \frac{2}{s^2+4}\right].$$

Having obtained expressions for the Laplace transforms of derivatives, we are now in a position to use Laplace transform methods to solve ordinary linear differential equations with constant coefficients.

Example 11. *Solve $\dfrac{dx}{dt} - 2x = 4$ given that $x(0) = 1$.*
Solution. *Assume that $L[x(t)] = X(s)$. Taking Laplace transform of both sides:*

$$\begin{aligned}
&L[\frac{dx}{dt}] - 2L[x] = L[4] \\
&sX(s) - x(0) - 2X(s) = \frac{4}{s} \\
&(s-2)X(s) - 1 = \frac{4}{s} \\
&X(s) = \frac{1}{s-2} + \frac{4}{s(s-2)}
\end{aligned}$$

Applying partial fraction method we get

$$\frac{4}{s(s-2)} = \frac{-2}{s} + \frac{2}{s-2}$$

so that

$$X(s) = \frac{1}{s-2} - \frac{2}{s} + \frac{2}{s-2} = \frac{3}{s-2} - \frac{2}{s}.$$

The second step is to take inverse Laplace transform of $X(s)$. Thus

$$\begin{aligned} x(t) = L^{-1}[X(s)] &= L^{-1}\left[\frac{3}{s-2} - \frac{2}{s}\right] \\ &= L^{-1}\left[\frac{3}{s-2}\right] - L^{-1}\left[\frac{2}{s}\right] \\ &= 3e^{2t} - 2 \end{aligned}$$

$\therefore x(t) = 3e^{2t} - 2$ *is the solution of the IVP.*

Example 12. *Solve* $\frac{dx}{dt} + 2x = 10e^{3t}$ *given that* $x(0) = 6$.

Solution. *Assume that* $L[x(t)] = X(s)$. *Taking Laplace transform of both sides we have*

$$\begin{aligned} L[\frac{dx}{dt}] + 2L[x] &= L[10e^{3t}] \\ sX(s) - x(0) + 2X(s) &= \frac{10}{s-3} \\ (s+2)X(s) - 6 &= \frac{10}{s-3} \\ (s+2)X(s) &= 6 + \frac{10}{s-3} \\ X(s) &= \frac{6}{s+2} + \frac{10}{(s+2)(s-3)} \end{aligned}$$

Applying partial fraction method we get

$$\frac{10}{(s+2)(s-3)} = \frac{-2}{s+2} + \frac{2}{s-3}.$$

So that

$$X(s) = \frac{6}{s+2} - \frac{2}{s+2} + \frac{2}{s-3} = \frac{4}{s+2} + \frac{2}{s-3}.$$

Taking inverse Laplace transform of $X(s)$, we get

$$\begin{aligned} x(t) &= L^{-1}[X(s)] \\ &= L^{-1}\left[\frac{4}{s+2} + \frac{2}{s-3}\right] \\ &= 4e^{-2t} + 2e^{3t}. \end{aligned}$$

$\therefore x(t) = 4e^{-2t} + 2e^{3t}$ *is the solution of the IVP.*

Example 13. *Solve the differential equation*

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = \sin t \quad t \geq 0$$

subject to the initial conditions $x(0) = 0$ *and* $\frac{dx}{dt}(0) = 0$.

Solution. *Taking the Laplace transforms*

$$L[\frac{d^2x}{dt^2}] + 6L[\frac{dx}{dt}] + 9L[x] = L[\sin t]$$

leads to the equation

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 6[sX(s) - x(0)] + 9X(s) = \frac{1}{s^2+1}$$

which on rearrangement gives

$$(s^2 + 6s + 9)X(s) = \frac{1}{s^2+1} + (s+6)x(0) + \dot{x}(0)$$

Incorporating the given initial conditions $x(0) = \dot{x}(0) = 0$ *leads to*

$$X(s) = \frac{1}{(s^2+1)(s+3)^2}.$$

Resolving into partial fractions gives

$$X(s) = \frac{3}{50}\frac{1}{s+3} + \frac{1}{10}\frac{1}{(s+3)^2} + \frac{2}{25}\frac{1}{s^2+1} - \frac{3}{50}\frac{s}{s^2+1}.$$

Taking inverse transforms, using the shift theorem, leads to the desired solution

$$x(t) = \frac{3}{50}e^{-3t} + \frac{1}{10}te^{-3t} + \frac{2}{25}\sin t - \frac{3}{50}\cos t.$$

Example 14. *Solve* $\ddot{x} - 2\dot{x} + 10x = e^{2t}$, *given* $x = 0, \dot{x} = 1$ *at* $t = 0$.

Solution. *Let* $X(s) = L[x(t)]$. *Taking the Laplace transforms*

$$L[\ddot{x}] - 2L[\dot{x}] + 10L[x] = L[e^{2t}]$$

leads to the equation

$$[s^2X(s) - sx(0) - \dot{x}(0)] - 2[sX(s) - x(0)] + 10X(s) = \frac{1}{s-2}$$

which on rearrangement gives

$$(s^2 - 2s + 10)X(s) = \frac{1}{s-2} + (s-2)x(0) + \dot{x}(0)$$

Incorporating the given initial conditions $x(0) = 0$ *and* $\dot{x}(0) = 1$ *leads to*

$$X(s) = \frac{\frac{1}{s-2} + 1}{s^2 - 2s + 10} = \frac{s-1}{(s-2)(s^2-2s+10)}.$$

Resolving into partial fractions gives $X(s) = \dfrac{A}{s-2} + \dfrac{Bs+C}{(s-1)^2+9}$ *we get* $A = \dfrac{1}{10}$, $B = -\dfrac{1}{10}$ *and* $C = 1$. *So that*

$$\begin{aligned} X(s) &= \frac{1}{10}\left[\frac{1}{s-2}\right] + \frac{-\frac{1}{10}s + 1}{(s-1)^2+9} \\ &= \frac{1}{10}\left[\frac{1}{s-2}\right] - \frac{1}{10}\left[\frac{s-1}{(s-1)^2+9}\right] + \frac{3}{10}\left[\frac{3}{(s-1)^2+9}\right] \end{aligned}$$

Taking inverse transforms, using the shift theorem, leads to the desired solution

$$\begin{aligned} x(t) &= \frac{1}{10}L^{-1}\left[\frac{1}{s-2}\right] - \frac{1}{10}L^{-1}\left[\frac{s-1}{(s-1)^2+9}\right] + \frac{3}{10}L^{-1}\left[\frac{3}{(s-1)^2+9}\right] \\ &= \frac{1}{10}e^{2t} - \frac{1}{10}e^t\cos 3t + \frac{3}{10}e^t\sin 3t \end{aligned}$$

$$\boxed{\therefore x(t) = \frac{1}{10}e^{2t} - \frac{1}{10}e^t\cos 3t + \frac{3}{10}e^t\sin 3t.}$$